ΘΕΜΑ

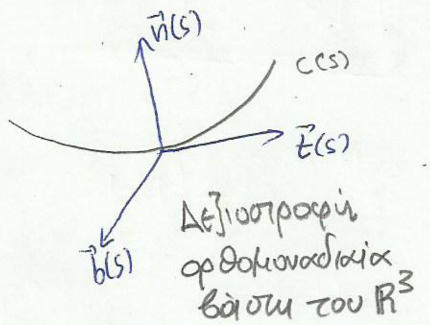
Ⓐ Δείξτε ότι κατά μήκος μιας καμπύλης c με φυσική παράμετρο το μήκος του τόξου:

i) $\dddot{c} = -k^2 \cdot \vec{t} + \dot{k}\vec{n} + k\cdot\tau\vec{b}$

ii) $[\dot{c}, \ddot{c}, \dddot{c}] = k^2 \cdot \tau$

Ⓑ Δείξτε ότι η στρέψη μιας καμπύλης με τυχαία παράμετρο t ώστε $k(t)>0$ είναι:

$$\tau = \frac{[c', c'', c''']}{\| c' \times c'' \|^2}$$



ΛΥΣΗ

Ⓐ i) Έχουμε: $\ddot{c} = \dot{\vec{t}} = k\vec{n}$

Έτσι, $\dddot{c} = \dot{k}\vec{n} + k\dot{\vec{n}} = \dot{k}\vec{n} + k\frac{d}{ds}(\vec{b}\times\vec{t}) =$

$= \dot{k}\vec{n} + k(\dot{\vec{b}}\times\vec{t} + \vec{b}\times\dot{\vec{t}}) \overset{frenet}{=}$

$= \dot{k}\vec{n} + k((-\tau\vec{n}\times\vec{t}) + \vec{b}\times k\vec{n}) =$

$= \dot{k}\vec{n} - k\tau(\vec{n}\times\vec{t}) + k^2(\vec{b}\times\vec{n}) =$

$= -k^2\vec{t} + \dot{k}\vec{n} + k\tau\vec{b}$.

ii) $\ddot{c}\times\dddot{c} = k\vec{n}\times(-k^2\vec{t} + \dot{k}\vec{n} + k\tau\vec{b}) =$

$= -k^3\vec{n}\times\vec{t} + \cancelto{0}{k\dot{k}(\vec{n}\times\vec{n})} + k^2\tau(\vec{n}\times\vec{b}) = k^3\vec{b} + k^2\tau\vec{t}$.

Έτσι, $(\dot{c}, \ddot{c}, \dddot{c}) = \langle \vec{t}, k^3\vec{b} + k^2\tau\vec{t} \rangle =$

$= \cancelto{0}{k^3\langle \vec{t}, \vec{b}\rangle} + k^2\tau\cancelto{1}{\langle \vec{t}, \vec{t}\rangle} = k^2\tau$.

(B). $\mathring{c} = \frac{dc}{ds} = \frac{dc}{dt} \cdot \frac{dt}{ds} = c' \cdot \mathring{t} \Rightarrow \boxed{\mathring{c} = c' \cdot \mathring{t}}$

$$\overset{\circ\circ}{c} = \frac{d}{ds}(c' \mathring{t}) = \frac{d}{ds}(c') \cdot \mathring{t} + \overset{\circ\circ}{t} \cdot c' = \frac{d}{dt}\left(\frac{dc}{dt}\right)\frac{dt}{ds} \cdot \mathring{t} + \overset{\circ\circ}{t} \cdot c' \Rightarrow$$

$$\Rightarrow \boxed{\overset{\circ\circ}{c} = (\mathring{t})^2 \cdot c'' + \overset{\circ\circ}{t} \cdot c'}$$

$$\overset{\circ\circ\circ}{c} = \frac{d}{ds}\left((\mathring{t})^2 \cdot c'' + \overset{\circ\circ}{t} \cdot c'\right) = \overset{\circ\circ\circ}{t} \cdot c' + \overset{\circ\circ}{t}\frac{d}{ds}(c') + 2\mathring{t}\overset{\circ\circ}{t} c'' + (\mathring{t})^2 \frac{d}{ds}(c'') \Rightarrow$$

$$\Rightarrow \boxed{\overset{\circ\circ\circ}{c} = \overset{\circ\circ\circ}{t} c' + 3\mathring{t} \cdot \overset{\circ\circ}{t} c'' + (\mathring{t})^3 \cdot c'''}$$

Έτσι, $(\mathring{c}, \overset{\circ\circ}{c}, \overset{\circ\circ\circ}{c}) = (c' \cdot \mathring{t})\left[\left((\mathring{t})^2 c'' + \overset{\circ\circ}{t} \cdot c'\right) \times \left(\overset{\circ\circ\circ}{t} c' + 3\mathring{t} \cdot \overset{\circ\circ}{t} c'' + (\mathring{t})^3 c'''\right)\right]$

$$\Rightarrow (\mathring{c}, \overset{\circ\circ}{c}, \overset{\circ\circ\circ}{c}) = \frac{(c', c'', c''')}{\|c'\|^6} \quad (1)$$

όμως, $k = \frac{\|c' \times c''\|}{\|c'\|^3}$ και $\frac{(\mathring{c}, \overset{\circ\circ}{c}, \overset{\circ\circ\circ}{c})}{k^2} = \frac{\cancel{k^2} \cdot \tau}{\cancel{k^2}}$

Άρα, $\tau = \frac{(\mathring{c}, \overset{\circ\circ}{c}, \overset{\circ\circ\circ}{c})}{k^2} = \frac{(c', c'', c''')}{k^2 \cdot \|c'\|^6} \Rightarrow$

$$\Rightarrow \tau = \frac{(c', c'', c''')}{\frac{\|c' \times c''\|^2}{\cancel{\|c'\|^6}} \cancel{\|c'\|^6}} = \frac{(c', c'', c''')}{\|c' \times c''\|^2}$$